MATH2060B: Analysis II: Home Project 2 **2 April 2021 at 5:00pm**

Important Notice:

♣ The answer paper **Must be submitted before 3 April 2021 at 5:00pm.**
♠ The answer paper MUST BE sent to the CU Blackboard.
✠ The answer paper **Must include your name and student ID.**

Answer ALL Questions

1. **(15 points)**

   Let f be a C^1-function defined on $(0,\infty)$. For $x \in \mathbb{R}$, let $[x]$ denote the greatest integer not greater than x.

   (i) Show that if a and b are positive integers with $a < b$, then

$$\sum_{k=a}^{b} f(k) = \int_a^b f(x)dx + \int_a^b f'(x)(x - [x] - \frac{1}{2})dx + \frac{f(a)+f(b)}{2}.$$

   (ii) Show that if $p \neq 1$, then

$$\sum_{k=1}^{n} \frac{1}{k^p} = \frac{1}{n^{p-1}} + p\int_1^n \frac{[x]}{x^{p+1}}dx.$$

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2. **(15 points)**

Let (f_n) be a sequence of bounded functions defined on $\mathbb{R}$. Suppose that $f(x) := \lim f_n(x)$ exists for all $x \in \mathbb{R}$.

(a) Show that

$$\lim_n \frac{f_1(x) + \cdots + f_n(x)}{n} = f(x)$$

for all $x \in \mathbb{R}$.

(b) If we further assume that (f_n) converges uniformly to f on $\mathbb{R}$, does it imply that the sequence $(\frac{f_1 + \cdots + f_n}{n})$ converges uniformly to f on $\mathbb{R}$?

3. **(20 points)**

Let f be a continuous function defined on $[a, b]$. Assume that the right derivative of f exists for every $x \in (a, b)$, that is, the limit $f'_+(x) := \lim_{t \to 0+} \frac{f(x+t) - f(x)}{t}$ exists.

(i) If $f(b) < f(a)$, we define a function $h : (f(b), f(a)) \to \mathbb{R}$ by

$$h(y) := \sup\{x \in (a, b) : f(x) > y\}.$$

Show that $f(h(y)) = y$ for all $y \in (f(b), f(a))$.

(ii) Let $D := \{x \in (a, b) : f'_+(x) > 0\}$. Show that if $(a, b) \setminus D$ is countable, then f is increasing.

$* * *$ **END OF PAPER** $* * *$